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Topology of a General Approximation System and Applications

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

In this paper we shall examine a generalization of the classical problem of approximation in normed linear spaces which we will call "vectorial approximation." Sections 1 and 2 contain a statement of the problem, examples, and some topological considerations used in the general theory. In Sections 3 and 4, the questions of existence, uniqueness and characterization are answered for special types of vectorial approximation.

Let X be a linear space of real-valued functions on $[a, b]$, Y an n-dimensional subspace of X, and let L_i be linear operators on X, for $i = 1, 2,..., k$. We denote by $|\cdot|_i$ an (abstract) norm on $X_i = \{L_i(x), x \in X\}$, $i = 1, 2,..., k$. A *vector-seminorm* $(\|\cdot\|, \leqslant\cdot)$ is defined as follows:

$$\| x \| = (| L_1(x)|_1, | L_2(x)|_2, ..., | L_k(x)|_k), \qquad \text{for} \quad x \in X,$$

and $\| x_1 \| \leqslant\cdot \| x_2 \|$, where $x_1, x_2 \in X$, if and only if $| L_i(x_1)|_i \leqslant | L_i(x_2)|_i$ for all $i = 1, 2,..., k$. If at least one L_i is the identity operator, $(\|\cdot\|, \leqslant\cdot)$ will be called a *vector-norm*. Given a basis $\{y_i\}$ of Y and an $x \in X$, we will say that $\alpha = (A_1, A_2, ..., A_n)$ or $y_\alpha = \sum_{i=1}^{n} A_i y_i$ is a *vectorial approximation* to x. We define $F_x(\alpha) \equiv \| x - y_\alpha \|$. We call α or y_α a *best vectorial approximation* to x if $F_x(\alpha)$ is a minimal point of the range of F, i.e., if there is no $\beta \in E^n$ such that $F_x(\beta) \leqslant\cdot F_x(\alpha)$ and $F_x(\beta) \neq F_x(\alpha)$.

For notational convenience, we shall use the absolute value symbol $|\cdot|$ to denote ordinary (real-valued) norms. The symbol $\|\cdot\|$ we will reserve for vector-valued norms and seminorms. If K is a subset of Euclidean n-space E^n, the symbol $F_x(K)$ will represent the set $\{F_x(\alpha) : \alpha \in K\}$. We will denote by $M(K)$ the set of minimal points of $F_x(K)$. Where the meaning is clear, we shall write M and $F(\alpha)$ instead of $M(E^n)$ and $F_x(\alpha)$. The *problem of vectorial* *approximation* is, roughly, the study of the equation $F(\alpha) = \mu$, for each $\mu \in M$. In way of justification of the terms "vector-norm" and "vector-seminorm" we note the following relations which are easy to verify:

$$\overbrace{(0, 0,..., 0)}^{k} \leqslant \cdot \| x \|,$$
$$(0, 0,..., 0) = \| 0 \|,$$
$$\| \theta x \| = | \theta | \, \| x \|,$$
$$\| x + y \| \leqslant \cdot \| x \| + \| y \|,$$

for all $x, y \in X$ and all real θ. Furthermore, if at least one L_i is the identity operator, then $\| x \| = (0, 0,..., 0)$ implies $x = 0$.

2. Examples and a General Theory of Vector-Norms

In the following examples [(a)–(e)] of vectorial approximation, let X be (quite arbitrarily) the space $C^7[0, 1]$, let Y be the class of polynomials of degree $\leqslant 5$, and let $x \in X$.

(a) Approximation of a function with respect to two norms.

$$| \cdot |_1 = \text{Chebychev (sup) norm},$$
$$| \cdot |_2 = L^3 \text{ norm},$$
$$L_1 = L_2 = I \,(\text{identity}).$$

(b) Approximation of a function and its derivative with respect to the Chebychev (sup) norm.

$$| \cdot |_1 = \text{sup norm},$$
$$| \cdot |_2 = \text{sup norm},$$
$$L_1 = I,$$
$$L_2 = \frac{d}{dx}.$$

(c) Approximation of a function in L^p norm, and its seventh derivative in L^3 + sup norm.

$$| \cdot |_1 = L^p \text{ norm},$$
$$| \cdot |_2 = (L^3 \text{ norm}) + (\text{sup norm}),$$
$$L_1 = I,$$
$$L_2 = \frac{d^7}{dx^7}.$$

(d) Approximation of a function f in the sup norm, of f' in L^3 norm, and of f in L^1.

$$|\cdot|_1 = \text{sup norm},$$
$$|\cdot|_2 = L^3 \text{ norm},$$
$$|\cdot|_3 = L^1 \text{ norm},$$
$$L_1 = I,$$
$$L_2 = \frac{d}{dx},$$
$$L_3 = I.$$

DEFINITION. Given $\epsilon \geqslant 0$, $g(x)$ is said to ϵ-interpolate $f(x)$ at $x_1, x_2, \ldots, x_m$ iff $\sum_{i=1}^{m} |f(x_i) - g(x_i)| \leqslant \epsilon$.

Observe that 0-interpolation is the usual interpolation.

(e) Constrained one-sided approximation, weighted ϵ-interpolation and L^p approximation. Require that $x(t) - y_\alpha(t) \geqslant 0$ on [0,1]; $y_\alpha(t)$ a polynomial of degree $\leqslant 5$.

$$|\cdot|_1 = L^p \text{ norm},$$
$$|\cdot|_2 = \text{sup norm},$$
$$|\cdot|_3 = L^1 \text{ norm},$$
$$L_1 = I,$$
$$L_2 = \sum a_i \hat{L}_i,$$
$$L_3 = I,$$

where a_i are positive weights, and $\hat{L}_i$ are the point functionals defined by $\hat{L}_i(f) = f(x_i)$.

(f) Application to model theory.

A standard electronic filter approximates, in the supremum norm, an ideal input-output characteristic. The number of the so-called lumped parameters (resistors, capacitors, etc.) equals the number of parameters of the approximation, while the way in which the lumped parameters are arranged (in series, in parallel, or in some other combination) corresponds to the type of approximation (linear, rational, etc.). In general, let S be an object we are interested in simulating by models $\mathscr{M}(\alpha)$, $\alpha \in J$ (α is a vector of parameters which varies over a set J). Let L_i be linear operators acting on the $\mathscr{M}(\alpha)$ and on S and let $|\cdot|_i$ $(i = 1, 2, \ldots, k)$ be a gauge of the goodness of fit of the i-th simulation feature. We use a minus sign "$-_i$" formally, to be interpreted in context, and the range of each $|\cdot|_i$ is ordered. For example, for some fixed i, let L_i be the color operator, let $L_i(S)$ = yellow, and let $\{L_i(\alpha), \alpha \in J\}$ = {orange, black, blue, green}. If it is further judged that |(yellow) $-_i$ (orange)$|_i$ = good, |(yellow) $-_i$ (blue)$|_i$ = |(yellow) $-_i$ (green)$|_i$ = medium, |(yellow) $-_i$ (black)$|_i$ = bad, we have such an ordering. Briefly, given S, a model space $\{\mathscr{M}(\alpha), \alpha \in J\}$, linear operators L_i on $S \cup \{\mathscr{M}(\alpha)\}$, interpretations of the binary operations $-_i$ $(i = 1, 2,..., k)$ which induce meaningful orderings on $|\cdot|_i$, then a *best model* $\mathscr{M}(\alpha)$ is one for which $F_s(\alpha)$ is a "best vectorial approximation." The realizability of a best model is synonymous to $M(J)$ being nonempty.

In what follows we prove some general theorems on vectorial approximation.

THEOREM 2.1. *$F_x(\alpha)$ is a continuous function of α.*

Proof. Let $\alpha = (A_1, A_2,..., A_n)$, $\beta = (B_1, B_2,..., B_n)$. The absolute value of the s-th component of the k-vector $F(\alpha) - F(\beta)$ is

$$\left| \left| L_s\left(x - \sum_{i=1}^{n} A_i y_i\right)\right|_s - \left| L_s\left(x - \sum_{i=1}^{n} B_i y_i\right)\right|_s \right|$$

$$\leqslant \left| L_s \sum_{i}^{n} (A_i - B_i)\, y_i \right|_s \leqslant \max_{1 \leqslant i \leqslant n} |A_i - B_i| \sum_{i}^{n} |L_s(y_i)|_s .$$

L_s and the y_i's are fixed; so $F_x(\alpha)$ is a continuous function of α.

COROLLARY. *If K is a compact subset of E^n, $M(K)$ is nonempty.*

Proof. $F(K)$ is also a compact subset of E^k; so the infima of chains of $F(K)$ relative to the usual, coordinate-wise partial ordering $\leqslant\cdot$ of E^k are assumed.

THEOREM 2.2. *$M(E^n) = M(I^n)$, for some cube $I^n \subset E^n$.*

Proof. Define an *extreme minimum* to be a point $(m_1^s, m_2^s,..., m_k^s)$ with the property that $\inf_{y \in Y} |L_s(x - y)|_s = m_s^s$. Note [1] that in case, for some s, L_s is the identity operator, there is a $y \in Y$ such that $|x - y|_s = m_s^s$ and, hence, the extreme minimum is a point of M.

For every s, $1 \leqslant s \leqslant k$, an extreme minimum $(m_1^s, m_2^s,..., m_k^s)$ does exist. Choose one. Now let $\mu = \max\{m_j^i : i = 1, 2,..., k, j = 1, 2,..., k\}$ and define $K_s = \max\{|L_s(x)|_s, \mu\}$. For $y \in E^n$ in the complement of

$$A = \{y : \|y\| \leqslant\cdot 3(K_1, K_2,..., K_k)\},$$

we have $|L_s(y)|_s > 3K_s$ for some s.

Therefore,

$$| L_s(x - y)|_s \geqslant | L_s(y)|_s - | L_s(x)|_s \geqslant 3K_s - | L_s(x)| > K_s \geqslant \max_i m_s^i.$$

So, $(\mu_1, \mu_2, ..., \mu_k) = F(y) \notin M(E^n)$. Hence, $M(E^n) \subset F(A)$. Now, the closed bounded set A is compact. Therefore, first, the infima of descending chains in $F(A)$ are assumed, and, secondly, there exists $I^n \supset A$ such that $M(E^n) = M(I^n)$. In what follows, we will write M instead of $M(E^n)$.

THEOREM 2.3.

(a) *$F^{-1}(\mu)$ is a convex subset of E^n, for each $\mu \in M$.*

(b) *M is closed. It consists of one point if and only if the extreme minima satsify $(m_1^i, m_2^i, ..., m_k^i) = (m_1^j, m_2^j, ..., m_k^j)$ for all i, j.*

Proof of (a). Let $F(\alpha) = F(\beta) = \mu$; then, for $0 \leqslant \theta \leqslant 1$,

$$F(\theta\alpha + (1 - \theta)\beta) \leqslant\cdot\ \theta F(\alpha) + (1 - \theta) F(\beta) = \mu.$$

Since $\mu \in M$, $F(\theta\alpha + (1 - \theta)\beta) = \mu$.

The proof of (b) is straightforward. Also, (b) can be strengthened as follows:

THEOREM 2.4. *Let $k = 2$ and let α_1 and α_2 be best (ordinary) approximations with respect to $| \cdot |_1$ and $| \cdot |_2$, respectively. If $F(\alpha_1) \neq F(\alpha_2)$ then M is a Jordan arc.*

Proof. See [2], p. 81.

The following theorem is a generalization of the classical case $k = 1$. Its complexity stems from the fact that, unlike the case $k = 1$, compact connected sets in E^k are generally hard to characterize. Let I^n be such that $M(I^n) = M(E_n)$ (see Theorem 2.2). For simplicity of notation we denote by S the compact, connected set $F(I^n) \subset E^k$.

THEOREM 2.5. *S is compact, locally connected, arcwise connected, and has a trivial $k - 1$ homotopy group, i.e., $\Pi_{k-1}(S) = 0$.*

Proof. Local connectivity follows from the Hahn–Mazurkiewicz Theorem. Arcwise connectivity follows from the fact that a Peano space is arcwise connected [7, p. 116].

To prove $\Pi_{k-1}(S) = 0$, consider the case $k = 2$. If $\Pi_1(S) \neq 0$, there exists a bounded component H of $E^2 - S$. This component contains a disc D because of the compactness of S (see Fig. 1).

Let p be the center of such a disc D. Now move D in the N-NE direction until the N-NE arc of D hits a first point $a \in S$. By the compactness of S,

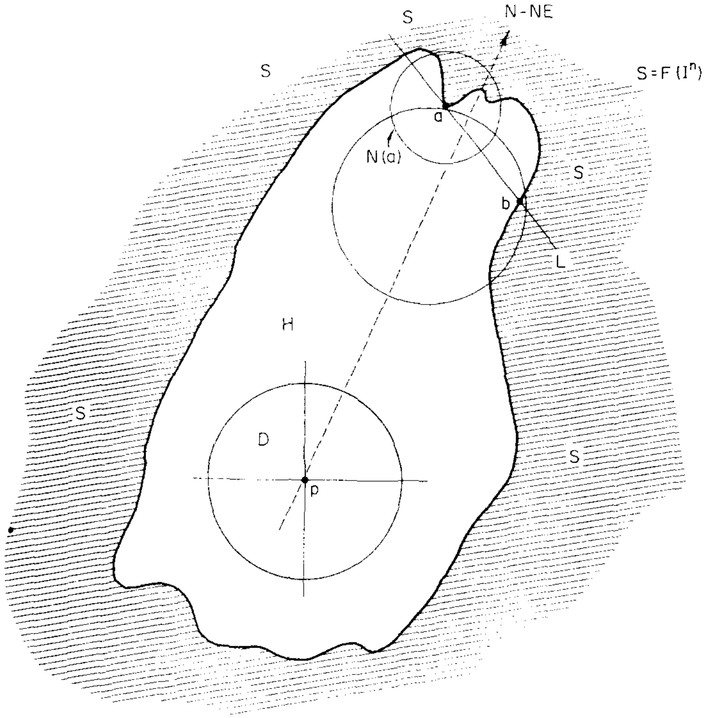

FIGURE 1

such a point exists. We assume, without loss of generality, that it is not the midpoint of the *N-NE* arc, for otherwise change slightly the preferred *N-NE* direction. From the point a draw a ray L of slope -1 until L hits S at b for the first time. It follows that there exists a neighborhood $N(a)$ in E^2 with the property that

$$N(a) \cap \left(\bigcup_{z \in L} \mathcal{O}(z) \right) \cap S = \varnothing, \tag{A}$$

where

$$\mathcal{O}(z) = \{x : x \in E^2, x \leqslant\cdot\, z\},$$

i.e., the points of $L \cap N(a)$ are "locally *S-W* accessible."

Now, by the continuity of F and the convexity of I^n, there exists a θ_0, $0 < \theta_0 < 1$, such that

$$F(\theta_0 \alpha + (1 - \theta_0)\beta) \in N(a) \cap S,$$

where

$$\alpha \in F^{-1}(a), \qquad \beta \in F^{-1}(b).$$

However, using (A) and the convexity of F (see Theorem 2.3), it follows that $F(\theta_0\alpha + (1 - \theta_0)\beta) \notin S$, contradicting the convexity of I^n.

The method of proof, for $k > 2$, is identical to that just described, for $k = 2$. For $k > 2$, choose H, again, as a bounded component of $E^k - S$; instead of the two-dimensional disc D use a k-dimensional open ball D^k, and in place of the N-NE arc, use the analogous subset of ∂D^k chosen in such a way that the corresponding L will have $k - 1$ negative directional cosines.

3. Chebychev Vectorial Approximation

Chebychev vectorial approximation has been studied by the author in [2]; however, for completeness, we include here a brief statement of results, without proofs.

Let $x \in C[a, b]$ be a function to be approximated and let Y be an n-dimensional Haar subspace of $C[a, b]$. That is, Y is an n-dimensional subspace of $C[a, b]$ such that zero is the only function in Y which vanishes at n distinct points of $[a, b]$. We shall assume that the functions $y_1(t),..., y_n(t)$ form a basis for Y. Let $w_1, w_2, ..., w_k$ be continuous and positive (weight functions) on $[a, b]$. We define $|\cdot|_i$ by

$$| z |_i = \sup_{t \in [a,b]} | w_i(t)\, z(t)|, \qquad i = 1, 2,..., k.$$

Define, furthermore, the set T_y "of critical" points of the approximation $y \in Y$ as follows:

$$T_{+i} = \{t \in [a, b] : w_i(t)(x(t) - y(t)) = | x - y |_i\},$$

$$T_{-i} = \{t \in [a, b] : w_i(t)(x(t) - y(t)) = -| x - y |_i\},$$

$$T_y = \left(\bigcup_{i=1}^{k} T_{+i}\right) \cup \left(\bigcup_{i=1}^{k} T_{-i}\right).$$

The existence of best vectorial approximations follows from the linearity of the approximating class Y (see Theorem 2.2). Proofs of characterization, uniqueness and the geometry of the minimal set M are given in [2, 3]. Observe that the following, perhaps surprising, results are generalizations of the standard theory of Chebychev approximation, i.e., $k = 1$ [1].

The function $F_x(\alpha) : E^n \to E^k$ and the minimal set M here are as defined in Section 1.

THEOREM 3.1 (Existence). *If μ is the infimum of any chain in $\{F_x(\alpha): \alpha \in E^n\}$, then there exists $\alpha \in E^n$ such that $F_x(\alpha) = \mu$.*

THEOREM 3.2 (Geometry). *Let $k = 2$ and let α_1 and α_2 be the best (ordinary) approximations with respect to $|\cdot|_1$ and $|\cdot|_2$, respectively. Then the minimal set M is a Jordan arc if and only if $\alpha_1 \neq \alpha_2$. If $\alpha_1 = \alpha_2$, M is a point.*

THEOREM 3.3 (Characterization). *Let $x \in C[a, b]$ and let $y \in Y$. Then the following statements are equivalent:*

(a) *y is a best vectorial approximation to x.*

(b) *The origin of Euclidean n-space E^n belongs to the convex hull of $\{\sigma(t)\hat{t} : t \in T_y\}$, where $\sigma(t) = -1$ if $t \in \bigcup_{i=1}^{k} T_{-i}$, $\sigma(t) = +1$ if $t \in \bigcup_{i=1}^{k} T_{+i}$ and $\hat{t} = (y_1(t), y_2(t),..., y_n(t))$.*

(c) *There exist $n + 1$ points $t_1 < t_2 < \cdots < t_{n+1}$ in T_y, satisfying $\sigma(t_i) = (-1)^{i+1} \sigma(t_1)$.*

THEOREM 3.4 (Uniqueness). *Each best vectorial approximation is unique, i.e., given $\mu \in M$, there is only one $\alpha \in E^n$ such that $F_x(\alpha) = \mu$.*

Observe that the uniqueness of Theorem 3.4 does not preclude the existence of *several* best vectorial approximations. In fact, Theorem 3.2 implies that, in general, there will be a whole Jordan arc of best vectorial approximations. Finally, note that much of this theory holds for more general approximating classes Y [2, 3].

A simple example which illustrates Theorems 3.1–3.4 is the following: Let $x(t) = t$ be approximated by constants $\{\alpha\}$, let $k = 2$, $w_1 = 1$, and

$$w_2(t) = \begin{cases} \dfrac{\delta - \epsilon}{\delta} t + \epsilon, & 0 \leqslant t \leqslant \delta, \\ t, & \delta \leqslant t \leqslant 1. \end{cases}$$

For small $\delta > 0$ and $\epsilon > 0$, it is easy to verify that the best approximations are those α satisfying $\frac{1}{2} \leqslant \alpha \leqslant -2 + 2\sqrt{2}$, and that the error of each best approximation exhibits vector-alternation. It is also seen that M here is the line segment joining the points

$$F(\tfrac{1}{2}) = (\tfrac{1}{2}, \tfrac{1}{2}) \qquad \text{and} \qquad F(-2 + 2\sqrt{2}) = (-2 + 2\sqrt{2}, 3 - 2\sqrt{2}).$$

4. L_2 VECTORIAL APPROXIMATION

Let $X = C[a, b]$ and let $(\cdot)_1$ and $(\cdot)_2$ be two inner products on X which induce, in the usual fashion, the norms $|\cdot|_1$ and $|\cdot|_2$, respectively. Let Y be an n-dimensional subspace of X, spanned by an orthonormal (with respect to $(\cdot)_1$) basis $z_1, z_2, \ldots, z_n$. Given $x \in X$ to be approximated, we define the two-dimensional vector-valued function $F(\alpha) = \| x - y_\alpha \|$ by

$$F(\alpha) = (\sqrt{(x - y_\alpha, x - y_\alpha)_1}, \sqrt{(x - y_\alpha, x - y_\alpha)_2}),$$

where $y_\alpha = \sum_{i=1}^n A_i z_i$. The problem here is to determine the minimal set M and, given $\mu \in M$, to determine all α's which satisfy the equation $F(\alpha) = \mu$. Observe that such α are generalizations of the classical Fourier coefficients. The equation $F(\alpha) = \mu$ will be shown to have a unique solution α. Furthermore, using the method of Lagrange multipliers, α will be given explicitly in terms of the solution of an algebraic equation in one unknown and of degree $2n$.

THEOREM 4.1 (Existence). *If μ is an infimum of the set $\{F(\alpha) : \alpha \in E^n\}$, then there exists an $\alpha \in E^n$ satisfying $F(\alpha) = \mu$. Hence $M \neq 0$.*

Proof. This follows from Theorem 2.2.

THEOREM 4.2 (Geometry). *M is either a point or a Jordan arc.*

Proof. This is a special case of Theorem 2.4.

THEOREM 4.3 (Solution of $F(\alpha) = \mu$). *Let $x \in C[a, b]$ and $\mu = (m_1, m_2) \in M$. The solution $\alpha = (A_1, A_2, \ldots, A_n)$ is unique and is given by*

$$A_k = -\frac{N'_k}{N_k} + (x, z_k)_1, \qquad k = 1, 2, \ldots, n,$$

where:

N_k is the determinant of the $n \times n$ matrix $\{N_{ij}\}$,

N_k' is the determinant of the $n \times n$ matrix $\{N'_{ij}\}$,

$$N'_{ij} = \begin{cases} N_{ij} & \text{if } j \neq i, \\ M_i & \text{if } j = i, \end{cases}$$

$$N_{ij} = b_{i1}b_{j1} + b_{i2}b_{j2} + \cdots + b_{in}b_{jn} \qquad \text{if } i \neq j,$$

$$N_{ii} = b_{i1}^2 + b_{i2}^2 + \cdots + b_{in}^2 + \lambda,$$

$$M_m = \sum_{i=1}^n (x, w_i)_2 \, b_{mi} - \sum_{i=1}^n (x, z_i)_1 \sum_{s=1}^n b_{ms} b_{is},$$

and where λ, w_i and b_{ij} will be defined in the proof.

Proof (by construction). Let $\{w_i\}$ be the Gram–Schmidt orthonormal sequence relative to $(\cdot)_2$, where $w_k = \sum_{i=1}^{k} a_{ki} z_i$. Let $\{b_{ki}\}$ be the inverse matrix of $\{a_{ki}\}$. Using standard Fourier theory on the expression $y \equiv x - \sum A_i z_i$, we have

$$((y, y)_1, (y, y)_2) = \Big((x, x)_1 - \sum_{i=1}^{n} (w, z_i)_1^2 + \sum_{i=1}^{n} (A_i - (x, z_i)_1)^2$$

$$\cdot (x, x)_2^2 - \sum_{i=1}^{n} (x, w_i)_2 + \sum_{i=1}^{n} (B_i - (x, w_i)_2)^2\Big),$$

where

$$B_i = \sum_{s=1}^{n} A_s b_{si}.$$

The remainder of the construction is a standard application of Lagrange multipliers. It involves minimizing $(y, y)_2$ as a function of α, where α satisfies the constraint $(y, y)_1 = m_1^2$. The resulting $\alpha(\lambda) = (A_1, A_2, \ldots, A_n)$ is that given in the statement of Theorem 4.3. λ is solved for by substituting the $\alpha(\lambda)$ in the constraint equation. This yields an algebraic equation for λ of degree $2n$ which may be solved by the various standard iterative methods. The details of this construction may be found in [4].

The uniqueness of α is seen as follows: From the above constructive proof it is clear that λ, and, therefore, α can have at most $2n$ values. By Theorem 2.3, $F^{-1}(\mu)$ is convex, and the result follows.

As a simple illustration of the above, we will compute M and the α satisfying $F(\alpha) = \mu \in M$, where F is given by

$$F(A_1, A_2) = \left\| t^2 - A_2 \frac{\sqrt{3}}{2} t - A_1 \frac{1}{\sqrt{2}} \right\| \quad \text{and}$$

$$(f, g)_1 \equiv \int_{-1}^{1} f(t) g(t)\, dt \quad \text{and} \quad (f, g)_2 \equiv \int_{-1}^{1} \frac{f(t) g(t)}{\sqrt{1 - t^2}}\, dt.$$

Observe that, here, $n = 2$, $x = t^2$ and the orthonormal vectors are

$$z_1 = \frac{1}{\sqrt{2}}, \quad z_2 = \frac{3}{2} t, \quad w_1 = \frac{1}{\sqrt{\pi}}, \quad w_2 = \frac{2}{\pi} t.$$

We evaluate

$$(t^2, z_1)_1 = \frac{\sqrt{2}}{3}, \quad (t^2, z_2)_1 = 0, \quad (t^2, t^2)_1 = \frac{2}{5}$$

$$(t^2, t^2)_2 = \frac{3}{8}\pi, \qquad (t^2, w_1)_2 = \frac{\sqrt{\pi}}{2}, \qquad (t^2, w_2)_2 = 0,$$

$$A_1 = -\frac{\frac{\pi}{2}\frac{\sqrt{2}}{3} - \frac{1}{\sqrt{2}}}{\lambda + (\pi/2)} + \frac{\sqrt{2}}{3}, \qquad A_2 = 0.$$

Substituting into the constraint equation $(A_1)^2 = m_1 - (8/45)$, we get for $F^{-1}(M)$ the set of all (A_1, A_2) given by

$$\begin{cases} A_1 = \left(\frac{\sqrt{2}}{3} - m_1 - \frac{8}{45}\right)^{1/2} \\ A_2 = 0, \end{cases} \qquad m_1 \in \left[\frac{8}{45}, \left(\frac{\pi}{8} + \frac{\sqrt{2\pi}}{3\sqrt{2}} - \frac{\sqrt{\pi}}{2}\right)^2\right].$$

Also, the coordinates of M are given by

$$\begin{cases} \sqrt{(t^2, t^2)_1} = \left(\frac{8}{45} + \left(A_1 - \frac{\sqrt{2}}{3}\right)^2 - A_2^2\right)^{1/2}, \\ \sqrt{(t^2, t^2)_2} = \left(\frac{\pi}{8} + \left(A_1 \frac{\pi}{2} - \frac{\sqrt{\pi}}{2}\right)^2 + \left(A_2 \frac{\sqrt{3\pi}}{2}\right)^2\right)^{1/2}, \quad (A_1, A_2) \in F^{-1}(M). \end{cases}$$

5. Comments and Unsolved Problems

Note that, perhaps surprisingly, much of the classical structure extends to the vectorial context in the cases of the best understood approximation spaces, namely Chebychev and L^2. One can, of course, generate a plethora of unsolved problems by specializing the norms and the operators. Two such interesting problems are

1. Characterize all best vectorial approximations with respect to a vector-norm composed of a supnorm and an L^2 norm.

2. Characterize all best vectorial approximations with respect to a vector-norm composed of the sup-norm of a function and the sup-norm of its derivative (see related work of P. J. Laurent, *Num. Math.* **10** (1967)).

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